

Counting covering cycles

G.A.T.F.da Costa ¹ and M. Policarpo
 Departamento de Matemática
 Universidade Federal de Santa Catarina
 88040-900-Florianópolis-SC-Brasil

Abstract

We compute the number of equivalence classes of nonperiodic covering cycles of given length in a non oriented connected graph. A covering cycle is a closed path that traverses each edge of the graph at least once. A special case is the number of Euler cycles in the non oriented graph. An identity relating the numbers of covering cycles of any length in a graph to a product of determinants is obtained.

1 Introduction

Let $G = (V, E)$ be a finite connected non oriented graph, V is the set of $|V|$ vertices, E is the set of edges with $|E|$ elements labelled $e_1, \dots, e_{|E|}$. The graph may have multiple edges and loops. Let's define a new graph from G as follows. Firstly, orient the edges. The orientation is arbitrary but fixed. Let G' be the graph with $2|E|$ oriented edges built from G by adding in the opposing oriented edges $e_{|E|+1} = (e_1)^{-1}, \dots, e_{2|E|} = (e_{|E|})^{-1}$, $(e_i)^{-1}$ being the oriented edge opposite to e_i and with origin (end) the end (origin) of e_i . In the case that e_i is an oriented loop, $e_{i+|E|} = (e_i)^{-1}$ is just an additional oriented loop hooked to the same vertex. A path in G is an ordered sequence of edges in G' , $(e_{i_1}, \dots, e_{i_N})$, $e_{i_k} \in E' = \{e_1, \dots, e_{2E}\}$, such that the end of e_{i_k} is the origin of $e_{i_{k+1}}$. In this paper, all paths are *cycles*. A cycle is a non-backtracking tail-less closed path, that is, the end of e_{i_N} coincides with the origin of e_{i_1} , subjected to the non-backtracking condition that $e_{i_{k+1}} \neq e_{i_k+|E|}$. In another words, a cycle never goes immediately backwards over a previous edge. Tail-less means that $e_{i_1} \neq e_{i_N}^{-1}$. The length of a cycle is the number of edges in its sequence. A cycle p is called periodic if $p = q^r$ for some $r > 1$, the integer r is the period of p , and q is a non periodic cycle. The cycle $(e_{i_N}, e_{i_1}, \dots, e_{i_{N-1}})$ is called a circular permutation of $(e_{i_1}, \dots, e_{i_N})$. Cycles that are circular permutations of one another are said to be equivalent. The cycle $(e_{i_N}^{-1}, \dots, e_{i_1}^{-1})$ is an inversion of $(e_{i_1}, \dots, e_{i_N})$. A cycle and its inverse are taken as distinct.

The number of cycles of given length in a non oriented graph G can be computed using the *the edge adjacency matrix* of G . This is the $2|E| \times 2|E|$ matrix T

¹g.costa@ufsc.br

defined as $T_{ij} = 1$, if end vertex of edge i is the origin of edge j and edge j is not the inverse edge of i ; $T_{ij} = 0$, otherwise. The number $\text{Tr} T^N$ counts backtrack-less and tail-less cycles of length N in G . The edge adjacency matrix is usually associated with the Ihara zeta function of a non oriented graph [9].

A cycle is called an edge covering cycle if every edge in the graph G is present at least once in the cycle. Therefore, if N is the length of an edge covering cycle, $N \geq |E|$. We are interested in counting equivalence classes of edge covering cycles in a non oriented graph. For that reason we consider graphs not having degree one vertices although proper subgraphs may have them. Restricted to this subgraph, say g , $\text{Tr} T_g^N$ will count only the backtrack-less and tail-less cycles of length N in g so that edges with 1-degree vertices may be deleted.

In section 2, we obtain a relation for counting the number of equivalence classes of non periodic covering cycles of given length in a non oriented graph. A special case is the number of Euler cycles in the non oriented graph. This was our initial motivation. But then we decided to consider covering cycles in general in order to find the determinantal identity in Theorem 2.4, inspired by an analogous but simpler identity associated to the Ihara zeta function of a graph. The identity resembles some identities associated with dynamical zeta functions [8]. The number of Euler cycles can be computed from one of the coefficients in the expanded form of this identity. We give examples in section 3.

2 Counting covering and Euler cycles

Some years back I asked Prof. Michael Krebs (State University of California) [6] if he knew how, along the lines of the combinatorial trace method [5], to compute the number of covering cycles of a given length in a graph. Few days later he replied and his ideas are encapsulated in Theorem 2.1.

Theorem 2.1 Denote by $\{G_k\}$ the set of graphs obtained from a graph G as the result of deleting k of its distinct edges, and by T_{G_k} the edge adjacency matrix of G_k . Set $G_0 := G$ and $G_{|E|}$ the empty graph with $\text{Tr} T_{G_{|E|}} = 0$. The number of covering cycles of length N in G is

$$\omega(N) = \sum_{k=0}^{|E|} (-1)^k \sum_{\{G_k\}} \text{Tr} T_{G_k}^N. \quad (2.1)$$

Proof: Apply the inclusion-exclusion principle to find the number of cycles that do not traverse edge e_1 , or e_2 , ..., or $e_{|E|}$. Then, subtract the result from $\text{Tr} T^N$ to get (2.1). \square

Remark 2.1. We have found in the literature some results that resembles very much (2.1) but associated to the counting of hamiltonian paths. They employ the vertex adjacency matrix. See [1], [4] and [7].

Remark 2.2. If $N < |E|$, $\text{Tr } T^N$ counts the non covering cycles of length N in the graph, but this is exactly what the other terms in the right hand side of (2.1) are doing in the formula so we get $\omega = 0$ in this case, as expected. If $N \geq |E|$, it can happen that $\omega(N) = 0$. For instance, in a bipartite graph there is no covering cycle of odd length, hence, only the non covering cycles are counted by $\text{Tr } T^N$ in this case but, again, this is exactly what the other terms together are doing in the formula so we get $\omega = 0$ in this case, too.

Theorem 2.2 *Let $\Theta(N)$ be the number of equivalence classes of all non periodic covering cycles of length N in the graph G . Then,*

$$\Theta(N) = \frac{1}{N} \sum_{g|N} \mu(g) \omega\left(\frac{N}{g}\right) \quad (2.2)$$

The number of Euler cycles in G is $\Theta(|E|)/2 = \omega(|E|)/2|E|$.

Proof: In the set of $\omega(N)$ covering cycles there is the subset with $N\Theta(N)$ elements formed by the non periodic covering cycles of length N plus their circular permutations and the subset with

$$\sum_{g \neq 1|N} \frac{N}{g} \Theta\left(\frac{N}{g}\right)$$

elements formed by the periodic covering cycles of length N (whose periods are the common divisors of N) plus their circular permutations. A covering cycle of period g and length N is of the form

$$p^g = (e_{k_1} e_{k_2} \dots e_{k_\alpha})^g$$

where $\alpha = N/g$, and p is a non periodic cycle so that the number of periodic cycles with period g plus their circular permutations is given by

$$\frac{N}{g} \Theta\left(\frac{N}{g}\right).$$

Hence,

$$\omega(N) = \sum_{g|N} \frac{N}{g} \Theta\left(\frac{N}{g}\right) \quad (2.3)$$

Möbius inversion formula gives the result. An Euler cycle is a cycle that covers each edge of G only once so the number of Euler cycles is given by $\Theta(|E|)$. Divide this number by 2 to avoid inversions. Notice that $\omega(|E|/g) = 0$ if $g > 1$ by Remark 2.2 so that $\Theta(|E|) = \omega(|E|)/|E|$. Of course, $\Theta(|E|)$ may be zero. This will be the case if the graph has vertices of odd degree. \square

Theorem 2.3 *Define*

$$h(z) := \sum_{N=1}^{\infty} \frac{\omega(N)}{N} z^N. \quad (2.4)$$

Then,

$$\prod_{N=1}^{+\infty} (1 - z^N)^{\pm \Theta(N)} = e^{\mp h(z)} = \prod_{k=0}^{|E|} \prod_{\{G_k\}} [\det(1 - zT_{G_k})]^{\pm (-1)^k} \quad (2.5)$$

$$= 1 \mp \sum_{i=1}^{+\infty} d_{\pm}(i) z^i \quad (2.6)$$

and

$$d_{\pm}(i) = \sum_{m=1}^i \lambda_{\pm}(m) \sum_{\substack{a_1 + 2a_2 + \dots + ia_i = i \\ a_1 + \dots + a_i = m}} \prod_{k=1}^i \frac{(\omega(k))^{a_k}}{a_k! k^{a_k}} \quad (2.7)$$

with $\lambda_+(m) = (-1)^{m+1}$, $\lambda_-(m) = +1$.

Proof: Compute the formal logarithm of the infinite product to prove the first equality in (2.5). Then, use (2.1) to get the second. (2.10) follows from Faà di Bruno relation for the exponential. See [2]. \square

Theorem 2.4 *Given a graph with $n_0 := |E|$ edges,*

- a) $nd_{\pm}(n) = \omega(n) \mp \sum_{k=1}^{n-1} \omega(n-k) d_{\pm}(k), n \geq n_0,$
- b) $d_-(n) = d_+(n) + \sum_{i=1}^{n-1} d_+(i) d_-(n-i), n \geq n_0,$
- c) $d_{\pm}(n) = 0, n \leq n_0 - 1,$
- d) $d_{\pm}(n) = \frac{\omega(n)}{n}, n_0 \leq n < 2n_0,$
- e) $d_+(n) \leq \frac{\omega(n)}{n}, 2n_0 \leq n < 3n_0,$
- f) $d_-(n) \geq \frac{\omega(n)}{n}, n \geq 2n_0,$
- g) $|d_+(n)| \leq d_-(n).$

The number of Euler cycles is $d_+(|E|)/2$.

Proof: See [3] for the proof of a) and b). The other items are an easy consequence of (2.7), a) and b) and Remark 2.2.

Remark 2.3. If $d_+(n) \geq 0$ for all n then a) implies that $d_+(n) \leq \omega(n)/n$ for all n . However, positivity does not hold for all graphs. For instance, the cycle graph which has one face and $|E| = |V| = n$ edges. For this graph, the double product of determinants in (2.5) is reduced to

$$\det(1 - zT) = (1 - z^{|V|})^2 = 1 - 2z^{|V|} + z^{2|V|} \quad (2.8)$$

which is the reciprocal of the Ihara zeta function of the graph. This seems to be the only case but a proof is lacking at the moment.

3 Examples

Example 1. G_1 , the non oriented graph with $R \geq 2$ edges hooked to a single vertex. Let's orient the edges counterclockwisely. The edge matrix for G_1 is the $2R \times 2R$ symmetric matrix defined on G'_1 ,

$$T_{G_1} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where A is the $R \times R$ matrix with all entries equal to 1 and B is the $R \times R$ matrix with the main diagonal entries equal to 0 and all the other entries equal to 1. This matrix has trace given by

$$\text{Tr } T_{G_1}^N = 1 + (R - 1)(1 + (-1)^N) + (2R - 1)^N, \quad N = 1, 2, \dots$$

Denote by G'_k the graph obtained from G_1 deleting k edges. Then,

$$\text{Tr } T_{G'_k}^N = 1 + (R - k - 1)(1 + (-1)^N) + (2R - 2k - 1)^N, \quad N = 1, 2, \dots$$

and

$$\omega_{G_1}(N) = \sum_{k=0}^{R-1} (-1)^k \binom{R}{k} [1 + (R - k - 1)(1 + (-1)^N) + (2R - 2k - 1)^N]$$

In the special case $R = 2$, the number of classes of nonperiodic covering cycles of length N is

$$\Theta_{G_1}(N) = \frac{1}{N} \sum_{g|N} \mu(g) \left((-1)^{\frac{N}{g}} - 2 + 3^{\frac{N}{g}} \right)$$

$\Theta_{G_1}(2)/2 = 2$ is the number of Euler cycles. Furthermore,

$$\begin{aligned} \prod_{N=1}^{+\infty} (1 - z^N)^{\Theta(N)} &= \prod_{k=0}^2 \prod_{\{G_k\}} [\det(1 - zT_{G_k})]^{(-1)^k} \\ &= \frac{(1 - 2z - 3z^2)}{(1 - z)^2} = 1 - \sum_{n=1}^{+\infty} 4(n - 1)z^n \end{aligned}$$

Example 2. G_2 , the bipartite graph with two vertices and three edges with the same orientation linking them. The edge matrix T_{G_2} , defined on G'_2 , is

$$T_{G_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix has the trace given by $\text{Tr} T_{G_2}^N = 0$ if N is odd and $\text{Tr} T_{G_2}^N = 4 + 2 \cdot 2^N$ if N is even, so $\omega_{G_2}(N) = 0$, if N is odd, and

$$\omega_{G_2}(N) = 2 \cdot 2^N - 8,$$

if N is even. In this case the number of nonperiodic cycles of length N is $\Theta_{G_2}(N) = 0$, if N is odd, hence, there are no Euler cycles, and

$$\Theta_{G_2}(N) = \frac{1}{N} \sum_{g|N, N/g \text{ even}} \mu(g) \left[2 \cdot 2^{\frac{N}{g}} - 8 \right],$$

We have

$$\begin{aligned} \prod_{N=1}^{+\infty} (1 - z^N)^{\Theta_{G_2}(N)} &= \frac{(1 - 4z^2)}{(z^2 - 1)^4} \\ &= 1 - \frac{1}{2} \sum_{N=2}^{+\infty} (N+2)(N+1)(N-1)z^{2N} \end{aligned}$$

4 Final comments

Let $G = (V, E)$ be a finite connected and directed graph with no 1-degree vertices, $|V|$ vertices, and $|E|$ edges $e_1, \dots, e_{|E|}$. The graph may have multiple edges and loops. A path in this case is given by an ordered sequence of oriented edges $(e_{i_1}, \dots, e_{i_N})$, $i_k \in \{1, \dots, |E|\}$, the end of e_{i_k} is the origin of $e_{i_{k+1}}$. Paths are always backtrack-less, tail-less and have no inverse. The number of cycles of a given length in G is given by the *directed vertex adjacency matrix* $A_d(G)$ [3]. Label the vertices of G , $1, 2, \dots, |V|$. Then, $(A_d)_{ij}$ is the number of edges directed from vertex i to vertex j . The number of cycles of length N in G is given by $\text{Tr} A_d^N$. Also, one can use the *directed edge adjacency matrix* of G to count cycles of a given length in the directed graph. This is the $|E| \times |E|$ matrix S defined as $S_{ij} = 1$, if end vertex of edge i is the start vertex of edge j ; $S_{ij} = 0$, otherwise. All the results in section 2 for non directed graphs hold in the directed case if one replaces the matrix T by S or A_d , $\text{Tr} S^N = \text{Tr} A_d^N$ [3]. In this case $\Theta(|V|)/2 = \omega(|V|)/2|V|$ is the number of hamiltonian cycles and we get the result in [4], section 3.6. Also, see [1] and [7].

Acknowledgments

G.A.T.F.C is deeply grateful to Professor Michael Krebs (State University of California) for his generous suggestion of the relation (2.1). Thanks to Prof. Asteroide Santana (UFSC) for help with latex commands and determinants.

References

- [1] E. T. Bax, Inclusion and exclusion algorithm for the hamiltonian path problem, Inf. Process. Lett. 47(4) 203-207 1993.
- [2] G. A. T. F. da Costa, J. Variance, Feynman identity: a special case revisited, Letters in Math. Phys. 73 (2005), 221-235.
- [3] G. A. T. F. da Costa, Generalized Witt identities, arXiv:1409.5767v2 [math.CO] (2014).
- [4] D. M. Cvetkovic, M. Doob, H. Sachs, Spectra of graphs, Academic Press, 1980.
- [5] K. Dsouza, M Krebs, A combinatorial trace method: counting closed paths to assay graph eigenvalues, Rocky Mountain J. Math vol 43 (2) 2013 469-478.
- [6] M. Krebs, E-mail correspondence, 21/11/2011.
- [7] S. N. Perepechko, A.N. Voropaev, The number of fixed length cycles in an undirected graph. Explicit formula in case of small lengths. Mathematical modelling and computational physics, Dubna, july 7-11, 2009.
- [8] D. Ruelle, Dynamical zeta functions and transfer operators, Notices Amer. Math. Soc. 49,887-895 (2002).
- [9] A. A. Terras, Zeta functions of graphs: a stroll through the garden, Cambridge University Press.